SIMULATION, ANALYSIS AND OPTIMALIZATION OF YARN WINDING AND UNWINDING PROCESSING


1. INTRODUCTION
In many textile processes, winding and unwinding of yarn is connected with rotation of yarn around a fixed axis, which leads to ballooning of the yarn. Each element of the ballooning yarn in these processes is loaded by a general system of forces in space, which creates continuous loading distributed along the length of moving yarn. The result of this action of force is a complex 3D curve. The surface of revolution generated by this rotating curve is a balloon, the shape, size and stability of which is determined by the boundary conditions of each particular examined process. The role played by yarn ballooning as a cause of tension variations in winding and unwinding processes has long been recognised in the textile industry. Indeed, in processes such as ring spinning, ring twisting and two-for-one twisting a yarn balloon is used as a tension regulation device. Up to now presented results [1], [2], [3] demonstrate the non-linear nature of this yarn ballooning phenomenon.

2. MATHEMATICAL MODEL FOR THE RING TWISTING PROCESS
In ring twisting or spinning, the movement of the yarn from delivery rolls of drafting mechanism to bobbin represents a complex motion. As shown in Fig. 1, it can be divided into four or five regions. It depends on using free or controlled balloon. Next we consider the balloon constrained with one limiter. Region I includes motion from roll delivery point to the yarn guide (pigtail), region IIa from pigtail to the balloon limiter, region IIb from the limiter to traveller. In region III the yarn is moving on the surface of limiter. In region IV there is yarn passage through the traveller and region V includes yarn motion from the traveller to wind-point on the bobbin. Ballooning yarn in this region creates a secondary balloon. The yarn rotation in regions IIa and IIb gives a rise to dynamic forces, which determine the level of tension produced in the yarn and its distribution along the yarn path. In recent papers [1], [6], [7], [8], the authors have described a computational model for ballooning yarn in ring spinning and gave some numerical results. The effect of the balloon control ring and the equations of traveller motion were incorporated into the new mathematical model for a ring twisting.

Equations of Motion of Yarn
In region I, the yarn path is practically a straight line. The yarn rotates about its axis to create twist and travels at the constant winding speed \( w \). Therefore, no dynamic forces are developed in this region.

In textile processes connected with the ballooning yarn (regions IIa, IIb) the determination of the dynamic equations in general shows that the problem involves
the solution of partial-derivative differential equations, which have to be integrated with the boundary conditions for a concrete yarn process. A solution based on the stationary assumption sufficiently suits for some of the operating processes and conditions. For regions IIA and IIB we consider the quasi-stationary case, where movement of the ring rail is very much slower than the winding speed of the yarn. That is the ring rail is nearly stationary, while yarn in these regions and traveller rotate at a constant speed. The solution of the periodic motion can be achieved by solving an appropriate sequence of stationary-balloon problem of varying balloon height or the winding radius. The ring spinning system is viewed from a state of rotating yarn with constant angular velocity. The investigated phenomenon is then described by system of ordinary non-linear differential equations.

Consider the case of yarn ballooning (Fig. 2), in which the bobbin axis is vertical and the yarn enters the system through a yarn guide situated above the bobbin on the axis of symmetry of the bobbin. The reference system commonly selected in balloon studies is cylindrical system \((O, r, \phi, z)\) with its origin \(O\) at the yarn guide, and axis \(z\) coinciding with the bobbin axis. This system rotates with an angular velocity \(\omega\) equal to the traveller’s rotating speed around the axis of the bobbin. The additional assumptions are made. The yarn is assumed to be perfectly flexible, linear elastic, and has uniform mass linear density. The air drag is assumed to act in the direction opposite to the component of the yarn velocity normal to the threadline, and it is assumed to be proportional to the square of the magnitude of this normal velocity component. The air drag coefficient is assumed to be constant along the whole length of the ballooning yarn. The effect of tangential air drag force on the yarn is neglected. This assumption has been investigated by Kothary and Leaf [2]. They found that, for yarn balloons, the effect of this force on the balloon profile and yarn tension is negligible.

Considering a small element of yarn \(ds\) the equilibrium equation is

\[ F + \frac{dT(T)}{ds} = 0, \quad (1) \]

where \(T\) is yarn tension force, \(t \left( \frac{dr}{ds}, \frac{d\phi}{ds}, \frac{dz}{ds} \right)\) is a unit vector tangent to the yarn directed toward the increasing \(s\) co-ordinate, \(s\) is length along the yarn. Force \(F\) is the sum of external forces (air drag and weight), plus centrifugal and Coriolis forces and inertial force of the sliding motion. The equilibrium equations of yarn in reference system \((O, r, \phi, z)\) can be written in the form, which was done by Migushov [3] as:

\[
\frac{d}{ds} \left( T_s \frac{dr}{ds} \right) - T_s \frac{d\phi}{ds} \frac{r^2}{ds} + \frac{\mu_0}{f(T)} \omega^2 r^2 + 2 \frac{\mu_0}{f(T)} w_T \omega \frac{d\phi}{ds} + F_{or} = 0
\]

\[
\frac{1}{r} \frac{d}{ds} \left( T_s r^2 \frac{d\phi}{ds} \right) - 2 \frac{\mu_0}{f(T)} w_T \omega \frac{dr}{ds} + F_{or} = 0
\]

\[
\frac{d}{ds} \left( T_s \frac{dz}{ds} \right) + \frac{\mu_0}{f(T)} g + F_{oz} = 0
\]

\[
T_s = T - \frac{\mu_0}{f(T)} w^2, \quad \frac{dr}{ds} = f(T), \quad \left( \frac{d\phi}{ds} \right)^2 + r^2 \left( \frac{d\phi}{ds} \right)^2 + \left( \frac{dz}{ds} \right)^2 = 1,
\]
where $F_\text{or}$, $F_\text{op}$, $F_\text{oz}$ are component of air drag force, $\mu_o$ is yarn mass per unit length, $l$ is unloading length of the yarn, $w$ is sliding velocity of the yarn, $g$ is gravity acceleration. The air drag force vector is given by:

$$F_a = -\frac{1}{2} C_n \rho D|v_n|v_n,$$

where $\rho$ is the air density, $D$ is the effective diameter of the yarn and $v_n$ is the vector of normal velocity component. Each element of the yarn is treated as a cylinder with its axis at an oblique angle to the airflow direction. The air drag coefficient $C_n$ is a function of the Reynolds number.

**Model of the Ring Limiter**

The controlled balloon with one ring limiter (Fig. 1) can be divided into three regions. The regions of ballooning curve Ia and Ib develop in the air and are subject to the same acting forces as free balloon. In the third region III the yarn is moving on the surface of ring limiter. The size of this region is very much smaller than total length of the ballooning yarn. Then the length of this region can be neglected and the ring limiter is modelled as a frictionless geometrical circle constraint (Fig 3) on the yarn path in the balloon.

The equilibrium equation of the yarn tension $T(s_{1-})$ on the upper side of the limiter the yarn tension $T(s_{1+})$ on the lower side of the limiter and the contact force between the yarn and the limiter must be considered at the frictionless point constraint on the limiter. Reaction force consists of a component of force $N$ normal to the limiter surface, and a frictional component $f N$. The vector of normal force is in meridian plane of balloon. In ring twisting the translational velocity is very much slower than rotating velocity of yarn. Thus the vector of frictional reaction on the limiter can be considered in the direction opposite to the rotating velocity of the yarn. It means that this reaction is acting in the tangent of the circle constraint. To do so we can express equilibrium equations at the point constraint $A_2$ in cylindrical coordinates, as

$$T(s_{1+}) r'(s_{1+}) + T(s_{1-}) r'(s_{1-}) - N_r = 0,$$

$$T(s_{1+}) r_{om} \phi'(s_{1+}) - T(s_{1-}) r_{om} \phi'(s_{1-}) - f N = 0,$$

$$T(s_{1+}) z'(s_{1+}) - T(s_{1-}) z'(s_{1-}) - N_z = 0.$$

Yarn tensions $T(s_{1+})$ and $T(s_{1-})$ are related by

$$T(s_{1+}) = T(s_{1-}) e^{\alpha},$$

where $\alpha$ is angle of contact, $f$ is coefficient of friction between yarn and limiter and $N_r$, $N_z$ are components of normal reaction.

**Equation of Motion for the Traveler**

The traveller is treated as a point mass with mass concentrated in the centre of mass of the traveller. Amontons’s law of friction is used to model the frictional force acting between the traveller and traveller ring, between the yarn and traveller, and between the yarn and control ring. Contact between the traveller and the ring is assumed to be at a single point. The effect of air drag on the traveller is neglected and the Euler equation is used to calculate the value of the yarn tension on either side of the traveller. The yarn between the traveller and the lay
point on the bobbin is assumed to follow a straight-line path lying in the plane of the traveller ring. Thus the forces acting on the simplified model of traveller are the yarn tension on the balloon side of the traveller, the yarn tension on the bobbin side of the traveller, normal reaction that ring exerts on the traveller in the direction normal to the ring, the frictional drag that the ring exerts on the traveller and gravity force of traveller. Afterwards the traveller’s equation of stationary motion can be expressed as follows:

$$T_n \sin \beta + f_T \cos \beta \cdot e^{-f_T \gamma} \left[ -f_T \frac{dr}{ds} + \left( f_T \frac{d\varphi}{ds} \right) + f_T \frac{dz}{ds} \right] - f_T m_T r_p \omega^2 + f_T m_T g = 0, \quad (6)$$

where $T_n$ is the yarn tension on the bobbin side of the traveller, $m_T$ is the mass of traveller, $f_T$ is the coefficient of ring - traveller friction, $f_T$ is the coefficient of friction between the yarn and the traveller, $\beta$ is the angle of the yarn between the traveller and the lay point makes with the radial direction, $\gamma$ is contact angle of the yarn, $\frac{dr}{ds}, \frac{d\varphi}{ds}, \frac{dz}{ds}$ is unit tangent vector to the yarn in the balloon at the traveller, $s_h$ is length of yarn in the balloon. Derivation of this equation has been described in detail in the previous paper [9].

**Boundary Conditions**

Referring to the cylindrical frame the following boundary conditions can be verified for the region IIa ($A_1A_2$) (Fig. 3). Geometrical conditions at the yarn guide ($s = 0$) are:

$$r(0) = 0, \quad \varphi(0) = 0, \quad z(0) = 0, \quad \varphi'(0) = 0. \quad (7)$$

If $s_1$ is length of yarn in the balloon between the yarn guide and the limiter, then the geometrical conditions at the ring limiter (point $A_2$) are:

$$r(s_1) = r_{om}, \quad z(s_1) = h_{om}, \quad (8)$$

where $r_{om}$ is radius of ring limiter, $h_{om}$ is distance of ring from yarn guide.

For the region IIb ($A_2A_3$), the equilibrium equations of the forces acting on the limiter must be considered. Boundary conditions at the ring limiter (point $A_2$) are:

$$r(s_1^+) = r(s_1^-), \quad \varphi(s_1^+) = \varphi(s_1^-), \quad z(s_1^+) = z(s_1^-),$$

$$T(s_1^+) r'(s_1^+) + T(s_1^-) r'(s_1^-) - N_r = 0,$$

$$T(s_1^+) \varphi'(s_1^+) - T(s_1^-) \varphi'(s_1^-) - f N = 0,$$

$$T(s_1^+) z'(s_1^+) - T(s_1^-) z'(s_1^-) - N_z = 0,$$

$$T(s_1^+) = T(s_1^-) e^{f \alpha},$$

$$\left[r'(s_1^+) \right]^2 + \left[r_{om} \varphi'(s_1^-) \right]^2 + \left[z'(s_1^-) \right]^2 = 1. \quad (9)$$

At the traveller ($s = s_2$) geometrical boundary conditions are:

$$r(s_2) = r_p, \quad z(s_2) = h, \quad (10)$$

where $r_p$ is radius of the traveller ring, $h$ is distance of the traveller ring from the yarn guide.

The final boundary condition is determined by the traveller’s equation of motion along the ring with two traveller reactions by relation (6):

$$T(s_2) = \frac{m_T r_p \omega^2 f_T}{(\sin \beta + f_T \cos \beta) e^{-f_T \gamma} + f_T r'(s_2) - r_p \varphi'(s_2) - f_T z'(s_2)}, \quad (11)$$

where the weight of the traveller is neglected.

**Numerical solution of the mathematical model**

In the stationary model, ring twisting is formulated as a boundary problem for a system of ordinary differential equations. The boundary problem can be solved using the multiple shooting method. The equations (2) together with boundary conditions are sufficient to determine the co-ordinates $r, \varphi, z$ of the ballooning threadline and yarn tension as functions of $s$ as well as the length of yarn in the balloon $s_1$ and $s_2$. The procedure is to solve equations (2) as an initial – value problem subject to initial conditions for the region IIa of threadline (7)
with trial values of $T(0)$ and $r'(0)$. Hamming’s modified predictor – corrector method is used to solve this initial – value problem, and the integration is stopped when $z(s_1) = h_{om}$, the prescribed distance of the limiter from the yarn guide. This determines $s_1$. After that the region IIb is solved as initial – value problem subject to initial conditions (9) with trial value of $r'(s_1)$. The integration is stopped when $z(s_2) = h$ (the prescribed balloon height). This determines $s_2$. A Newton – Raphson scheme is used to adjust the values $T(0)$, $r'(0)$ and $r'(s_1)$ until the other boundary conditions at the ring limiter (point $A_2$) and at the ring traveller (point $A_3$) are satisfied.

**Analysis of the Ring Twisting Process**

It is known from practical observations that when spinning with a traveller of the same mass, the maximum diameter of the balloon remains constant approximately with increasing r. p. m. of the traveller. This phenomenon is analysed by means of numerical simulation with the operating parameters and boundary conditions of the specific twisting frame. The dependence of the maximum balloon radius and of the tensile force of the balloon upon the r. p. m. of the traveller have been calculated under the following assumptions: all friction coefficients in the twisting frame are constant for the chosen interval of r. p. m., which to be analysed, the number of twists per one length unit is constant, winding diameter on the bobbin is constant and weight of the traveller is neglected.

The dependence has been analysed both for a free balloon and for a controlled balloon with one balloon limiter. The mass of the traveller has been chosen in such a way that at the speed of the traveller 3600 r. p. m., the maximum radius of the balloon without limiter might be $r_{max} = 85$ mm. The r. p. m. of the traveller has been increased gradually, at a constant mass of the same, looking for the maximum diameter of the balloon. The results of the simulation are shown in the Fig. 4. The numerical solution implies that the maximum balloon radius is constant at increasing r. p. m. of the traveller (diagram No. 4). The dependence of the tensile force in the ballooning yarn is of quadratic character (diagrams Nos. 1, 2). If including a

![Graph showing the dependence of the yarn tension force and maximal balloon radius upon the angular traveller speed](attachment:plot.png)

*Fig. 4: Dependence of the yarn tension force and maximal balloon radius upon the angular traveller speed*
balloon limiter, with the mass of the traveller always the same, the maximum radius of the balloon is reduced considerably (diagram No. 5). At the same time, the tensile force is reduced slightly (diagram No. 2) in comparison with the course of the tensile force in a free balloon (diagram No. 1). Both these changes are brought out by the fact that the centrifugal forces in the balloon are absorbed partially by the limiter. The reduction of the size of the balloon with the limiter allows to reduce the mass of the traveller considerably, e. g. to such a value that the maximum radius may achieve the original value $r_{\text{max}} = 85 \text{ mm}$ (diagram No. 6). As a consequence, the tensile force in the balloon drops down considerably (diagram No. 3), allowing to increase the r. p. m. of the spindles of the ring spinning frame in practice, providing other conditions are met, too. For example, an equal value of the tensile force is achieved at the speed of 4000 r. p. m. with a free balloon, and at the speed of 6200 r. p. m. with a controlled one.

**Optimisation of the Ring Twistin Frame**

The aim of the ring frame twisting line optimisation is to propose fundamental geometric and operational parameters of the twisting machine for certain range of mass linear density of twisted glass silk. The following parameters are concerned: balloon height, ring diameter, balloon limiter position and spindle speed. In terms of the number of doffs per time unit, it is advantageous to maximise the yarn winding volume on a bobbin, which would reduce the time necessary to replace full bobbins with empty ones. With the ring diameter given, the balloon height determines the winding volume on the bobbin. For this reason, balloon height maximisation with ring diameter given was chosen as a goal of the optimisation.

The following limiting criteria have to be considered for the twisting line optimisation: maximum velocity of traveller along the ring, allowable maximum balloon diameter, permitted mechanical stress of glass silk. Angular traveller speed and ring diameter are interdependent according to the relation given for maximum allowable circumferential velocity of the traveller: $v = r_p \omega = \text{const.}$ By using a mathematical model of ballooning yarn, dependencies of the balloon height and the limiter position upon the traveller speed and, at the same time, upon the traveller ring diameter are calculated for three values of glass silk mass linear density (Fig. 5). The optimisation is made with the following parameters:

- maximum balloon radius - $r_{\text{max}} = 1.2 r_p$,  
- diameter of winding equals to maximum bobbin diameter,  
- the maximum traveller velocity - $v_b = 40 \text{ m.s}^{-1}$,  
- balloon height corresponds to lower position of ring rail.
The ring twisting line optimisation presented enables to propose optimum basic parameters of twisting line for the required volume of winding or for the chosen r.p.m. of spindles.

3. NONSTATIONARY MATHEMATICAL MODEL

Schematic configuration of over-end unwinding is in Fig. 6. The yarn first slides across the package surface from unwind point L until its tangent angle $\alpha$ is right for it to lift off the package surface and to do away into the balloon. The point where this occurs is called the lift-off point M. The motion of a ballooning unwinding yarn is periodic as the unwinding point moves backwards and forward along the length of the package surface. The solution of the periodic motion can be achieved by solving an appropriate sequence of stationary-balloon problem of varying balloon height or the bobbin’s winding radius [5]. However, it is satisfactory only in cases, where the unwinding process is slow. Balloon shape is subjected to the rapid variations that occur during unwinding yarn at high speed.

Let’s consider the case of ballooning yarn during unwinding process according to Fig. 6. Let’s set the Cartesian coordinate system $(O, \xi, \eta, \zeta)$ with its origin O at the guide eye and the $\zeta$ axis identical with the axis of the bobbin. A discrete model is used for description of the ballooning yarn. The yarn is simulated by a chain of finite one-dimensional mass elements of the same length $l$ and the same weight. For simplicity let’s consider the number of elements $n = 6$. Let’s presume the yarn to be ideally flexible and inextensible. The yarn mass per unit length is constant throughout its length.

According to Fig. 7, vectors of elements $v_1, v_2 ... v_6$ are defined by the following relation

$$v_j = (a_j, b_j, c_j),$$

where $a_j, b_j, c_j$ are local co-ordinates of the $j^{th}$ element. The vector of the end point of the $j^{th}$ element is determined by the following equation

$$r_j = M + v_1 + v_2 + ... + v_j = M + (x_j, y_j, z_j).$$

Let’s parametrize the points of each element of the considered chain as follows. For example, function $h_3(u) = r_2 + v_3\left(\frac{1}{2} + u\right)$ represents the points of element 3 for $u \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. The kinetic energy of element $du$ in point $h_j(u)$ is determined by the relation

$$\frac{1}{2} \nu l [r_{j-1} + \dot{r}_j\left(\frac{1}{2} + u\right)]^2,$$

where $\nu$ is the yarn mass density. The equation of kinetic energy for the system of points $h_1(u), h_2(u), ..., h_6(u)$ can be represented as follows:

$$\frac{1}{2} \nu l \sum_{j=1}^{6} (r_{j-1} + \dot{r}_j\left(\frac{1}{2} + u\right))^2 du = T(u) du,$$
where the velocity of lift-off point from the surface of the bobbin M is \( \dot{r}_0 = 0 \). After differentiation of the kinetic energy by \( a_1, a_2, \ldots, a_6 \) we will get the equation:

\[
T_a = \nu l Q a,
\]

where \( T_a = \left[ \frac{\partial T}{\partial a_1}, \ldots, \frac{\partial T}{\partial a_6} \right]^T \) and \( Q \) is a symmetric matrix, where \( q_{ii} = 6 - i + \frac{1}{3} \), \( q_{ij} = 6 - i + \frac{1}{2} \) for \( j < i \). Similarly we will get equations for coordinates \( b \) and \( c \):

\[
T_b = \nu l Q b, \quad T_c = \nu l Q c.
\]

Potential energy \( V \) of the chain of elements is defined by the relation

\[
V = \nu l g \left[ (6 - \frac{1}{2})k_1 + (5 - \frac{1}{2})k_2 + \ldots + (1 - \frac{1}{2})k_6 \right],
\]

where \( g \) is gravity acceleration. Then functional \( G \) defined by the relation

\[
G = T - V + \sum_{j=1}^{6} \lambda_j v_j^2
\]

is maximized under the condition \( |v_j| = l^2 \), for \( j = 1, 2, \ldots, 6 \). Then we will get Lagrange's equations:

\[
v l Q a = A \lambda, \quad v l Q b = B \xi, \quad v l Q c = C \zeta
\]

where \( A = \text{diag}(a_1, a_2, \ldots, a_6) \) is a diagonal matrix, the \( j \)-th element on the diagonal is \( a_j \). Similarly for \( B, C \).

In addition let's include external forces in the mathematical model, such as the air drag. If generally the forces \( (F_1, F_2, \ldots) = F \) in original coordinates \( (x, y, z) \) enter into the system of equations (8), for the new co-ordinates \( (a_j, b_j, c_j) \) we will get forces \( x_a^j F, x_b^j F, x_c^j F \), where \( x_a, x_b, x_c \) are lower triangular matrices with elements equal to 1. If the forces act in points \( u \) of the element, the force in coordinates \( a \) (similarly \( b, c \)) will be:

\[
\xi_a^j F(u) du, \quad \text{where} \quad \xi_a = x_a + \left( u - \frac{1}{2} \right) E = \eta_b = \zeta_c \quad (E \text{ is a unit matrix}).
\]

The air drag force acting on the yarn element \( j \) we will reduce to the force acting in the centre of the element. Let's decompose the velocity vector of the centre of the element to the component parallel with the element and the component orthogonal to the element. The orthogonal (normal) component for element \( j \) will be named \( \tilde{h}_j^{(n)} \). In the model we will consider only the component of air drag in the normal direction, the dimension of which is proportional to the square of the normal component of velocity and has the opposite direction than this component. The air drag coefficient \( \beta \) is assumed to be constant along the whole length of the ballooning yarn. The effect of tangential air drag force on the yarn is neglected. Then aerodynamic force for element \( j \) in basic coordinates is:

\[
\beta \text{ res}_j = -\beta \tilde{h}_j^{(n)} |\tilde{h}_j^{(n)}|.
\]

For coordinates \( a, b, c \) we will get

\[
\begin{align*}
\text{(resa)}^T &= \begin{bmatrix} d_1 \xi_1^{(n)} & d_2 \xi_2^{(n)} & \ldots & d_6 \xi_6^{(n)} \end{bmatrix} \xi_a, \\
\text{(resb)}^T &= \begin{bmatrix} d_1 \eta_1^{(n)} & d_2 \eta_2^{(n)} & \ldots & d_6 \eta_6^{(n)} \end{bmatrix} \eta_b, \\
\text{(resc)}^T &= \begin{bmatrix} d_1 \zeta_1^{(n)} & d_2 \zeta_2^{(n)} & \ldots & d_6 \zeta_6^{(n)} \end{bmatrix} \zeta_c,
\end{align*}
\]

where
Then we consider Amontons’s law of friction to model the frictional force acting between the yarn and the surface of the bobbin. At sliding of the yarn on the surface of the cylindrical bobbin the normal reaction affects the yarn. Its point of action will be selected in the end point of the element. In this point \((x_i, y_i, z_i)\) the reaction \((x_i, y_i, 0)\ c_i v_i\) will be acting. So, in coordinates \(x_i\) we will get the following force vector

\[
(x_i c_i v_1, x_i c_i v_2, \ldots, x_i c_i v_k) = \mathbf{c}_i v^T \mathbf{X}_k,
\]

where \(\mathbf{X}_k = \text{diag}(x_1, \ldots, x_k)\). Here \(k\) is the number of yarn elements moving on the surface of the bobbin, for which \((\zeta_i > \zeta_0) \wedge (\xi_i^2 + \eta_i^2 = r_i^2)\), where \(\zeta_0\) is the coordinate of points of the upper spool flange, \(r_i\) is the radius of the cylindrical bobbin. For coordinates \(a\) we will then get \(\mathbf{x}_i^T \mathbf{X}_a c_i v\). Let’s presume that the friction drag of yarn winding acts against the speed vector of the end point of the element and is proportional to the friction coefficient \(\alpha\) and the normal reaction. Let’s mark

\[
\mathbf{X}_1 = \mathbf{x}_i^T (\mathbf{X}_k - \alpha \mathbf{X}_k^0),
\]

where \(\mathbf{X}_k^0 = \text{diag}(x_i^0, x_i^0, \ldots, x_i^0)\). Similarly we will set \(\mathbf{Y}_1\) and \(\mathbf{Z}_1 = \mathbf{x}_i^T (-\mathbf{Z}_k^0)\), where \((\mathbf{x}_i^0, \mathbf{y}_i^0, \mathbf{z}_i^0)\) are components of the unit velocity vector \((x_i, y_i, z_i)\). Matrices \(\mathbf{X}_i, \mathbf{Y}_i, \mathbf{Z}_i\) are supplemented with zero rows up to the total dimension \((n \times k)\) according to the stated example.

In the end we will add to the right side of the system of equations (8) a term representing the tensile force in the guide eye, i.e. the force acting in the end point of \(6\)th element. It is the term \([0, 0, \ldots, 0] f_i^T / f_i^T \mathbf{x}_i^T = f_i^T \mathbf{t}_i\), analogously for components \(b\) and \(c\), where \(\mathbf{t}_i^T = (1, 1, \ldots, 1)\).

So the final form of the system of equations is:

\[
\begin{align*}
\mathbf{v}^T \mathbf{Q} \ddot{a} &= A \lambda + f_a \mathbf{t} + \beta \mathbf{res}_a + \lambda_1 \mathbf{c}_i v \\
\mathbf{v}^T \mathbf{Q} \ddot{b} &= B \lambda + f_b \mathbf{t} + \beta \mathbf{res}_b + \lambda_2 \mathbf{c}_i v \\
\mathbf{v}^T \mathbf{Q} \ddot{c} &= C \lambda + f_c \mathbf{t} + \beta \mathbf{res}_c + \lambda_3 \mathbf{c}_i v - \mathbf{v}^T \mathbf{g} \mathbf{p},
\end{align*}
\]

where \(\mathbf{p} = \begin{pmatrix} 6 - \frac{1}{2}, 5 - \frac{1}{2}, \ldots, 1 - \frac{1}{2} \end{pmatrix}\).

Vectors \(\mathbf{f} = \begin{pmatrix} f_a \\ f_b \\ f_c \end{pmatrix}\), \(\mathbf{c}_i v\), \(\lambda\) are determined by solution of system of linear algebraic equations:

\[
\mathbf{S} \lambda + (\mathbf{A} \mathbf{Q}^{-1} \mathbf{X}_1 + \mathbf{B} \mathbf{Q}^{-1} \mathbf{Y}_1 + \mathbf{C} \mathbf{Q}^{-1} \mathbf{Z}_1) \mathbf{c}_i v + (\mathbf{A} \mathbf{Q}_{x_i}, \mathbf{B} \mathbf{Q}_{x_i}, \mathbf{C} \mathbf{Q}_{x_i}) \mathbf{f} =
\]

\[
- \beta (\mathbf{A} \mathbf{Q}^{-1} \mathbf{res}_a + \mathbf{B} \mathbf{Q}^{-1} \mathbf{res}_b + \mathbf{C} \mathbf{Q}^{-1} \mathbf{res}_c) + \mathbf{v}^T \mathbf{g} \mathbf{Q}^{-1} \mathbf{p} - \mathbf{v}^T \mathbf{w},
\]

\[
(\mathbf{X}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{A} + \mathbf{Y}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{B}) \lambda + (\mathbf{X}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{X}_i + \mathbf{Y}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{Y}_i) \mathbf{c}_i v +
\]

\[
(\mathbf{X}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{c}_i v) \lambda + (\mathbf{X}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{A} + \mathbf{Y}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{B}) \mathbf{c}_i v +
\]

\[
(\mathbf{X}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) + \mathbf{Y}_k \mathbf{L}_k \mathbf{Q}^{-1}(k)) \mathbf{c}_i v + (\mathbf{X}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{X}_i + \mathbf{Y}_k \mathbf{L}_k \mathbf{Q}^{-1}(k) \mathbf{Y}_i) \mathbf{c}_i v.
\]
Optimalizace vlastností strojů a pracovních procesů

\[ \begin{bmatrix} X_k \cdot L_k \cdot Q^{-1}(k) t, Y_k \cdot L_k \cdot Q^{-1}(k) t, 0 \end{bmatrix} = -X_k \cdot L_k \cdot Q^{-1}(k) \ \text{resa} - Y_k \cdot L_k \cdot Q^{-1}(k) \ \text{resb} - \nu l \begin{pmatrix} x_k^2 + y_k^2 \\ \vdots \\ x_k^2 + y_k^2 \end{pmatrix} \]

\[
\begin{bmatrix}
q_r^T \\
q_r^T \\
q_r^T \\
q_r^T \\
q_r^T \\
q_r^T \\
\end{bmatrix} = A
\begin{bmatrix}
\begin{bmatrix} X_1 \\
\end{bmatrix} \\
\begin{bmatrix} Y_1 \\
\end{bmatrix} \\
\begin{bmatrix} L_k \\
\end{bmatrix} \\
\begin{bmatrix} Z_k \\
\end{bmatrix} \\
\begin{bmatrix} Y_k \\
\end{bmatrix} \\
\begin{bmatrix} X_k \\
\end{bmatrix} \\
\end{bmatrix}
\lambda + \begin{bmatrix}
\begin{bmatrix} q_r \ \text{resa} \ \\
\end{bmatrix} \\
\begin{bmatrix} q_r \ \text{resb} \ \\
\end{bmatrix} \\
\begin{bmatrix} q_r \ \text{resc} \ \\
\end{bmatrix} \\
\begin{bmatrix} q_r \ \text{p} \ \\
\end{bmatrix} \\
\end{bmatrix} + gvl
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix} + vl \ PZ, \\
\end{bmatrix}
\]

where \( S = AQ^{-1} A + BQ^{-1} B + CQ^{-1} C \), \( w = [v_1^2, v_2^2, \ldots, v_n^2]^T \), \( \dot{v}_j = a_j^2 + b_j^2 + c_j^2 \), \( q_s \) is the sum column of matrix \( Q^{-1} \), \( L_k \) is the lower triangular matrix \((k \times k)\) with elements 1, \( Q^{-1}(k) \) is the matrix of first \( k \) rows of matrix \( Q^{-1} \), \( q_r^T \) is the row sum of matrix \( Q^{-1} \), \( q_r^T \) is the sum of elements of vector \( q_r^T \). \( PZ \) is optional acceleration of point 6.

Equation (13) is a result of the condition \( a_j^2 + b_j^2 + c_j^2 = l_j^2 \), after double differentiation:
\[ \ddot{a}_j a_j + \ddot{b}_j b_j + \ddot{c}_j c_j + a_j^2 + b_j^2 + c_j^2 = 0, \ j = 1,2, \ldots, 6. \]

Equation (14) results from the condition \( \dot{x}_i x_i + \dot{y}_i y_i = 0 \) (the velocity is normal to the cylindrical surface), after differentiation \( \dot{x}_i x_i + \dot{y}_i y_i + \ddot{x}_i^2 + \ddot{y}_i^2 = 0, \ i = 1, ..., k. \) Equation (15) expresses the relation that acceleration of point 6 corresponds to the specified \( PZ \) vector.

In addition we will determine the tension force in point M. In this model we consider only the vertical component. According to the review: the sum of forces affecting the element is equal to the sum of acceleration multiplied by mass we get the differential equation:
\[
f_{y,1} - f_{y} = vl g + \beta (\text{resa})_j + (Z_1 \ \text{civ})_j = vl \begin{bmatrix} 1, \ 1, \ \frac{1}{2} \ 0, \ 0, \ \ldots \end{bmatrix} \ddot{c} \]

Fig.8: Graphical output from software (Projections of ballooning yarn on xz and xy planes and balloon profile).
The initial condition $f_0 = f$ is the calculated tensile force. Reaction in point M is $f_0$. If the reaction exceeds the specified threshold value, point M is released, becomes point 1 and the next point on the bobbin gets to the place of M.

In the end we will describe the algorithm of solution:
The basis is the system of differential equations (12), which we will convert to the system of $1^{st}$ degree. The state variables are:

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n, \dot{a}_1, \dot{a}_2, \ldots, \dot{a}_n, \dot{b}_1, \dot{b}_2, \ldots, \dot{b}_n, \dot{c}_1, \dot{c}_2, \ldots, \dot{c}_n$$ (17)

Let these variables be given. First we will specify $\text{resa}, \text{resb}, \text{resc}$ according to explicit formulas, then we solve vectors $\text{civ} (\dim k), f (\dim 3), \lambda (\dim n)$ from the system of algebraic equations (13), (14), (15). By solution of the differential equation (16) we will get reaction $f_0$ in point M. If $|f_0|$ is greater than friction drag of the yarn, the yarn is lifted off the package surface. Then follows the next step of solution of the system of differential equations (12). If $\zeta_6 \geq 0$, then point 5 comes instead of point 6.

For the mathematical model and the proposed algorithm of solution a computer program was created with graphical outputs. Fig. 8 shows examples of results of the yarn unwinding simulation from the cross-wound cylindrical bobbin.

**Preliminary experiments, scanning the balloon shape**

A fast video camera was used to record the balloon shapes. In order to scan 3D images, it would be necessary to have two synchronised scanning systems positioned at an angle of 90° from each other. This is both technically and financially very demanding. There was, however, only one system available. Therefore, a mirror was used. The mirror was rotated 45° from the scanning plane and it scanned the images of the balloon rotated by 90°. The scanned images of the balloon therefore contained the direct image of the balloon and the image rotated by 90°, i.e. the mirror image. This solution is quite satisfactory, but it is necessary to keep in mind the fact that the mirror image has a smaller view angle and thus
a smaller image of the same object. It is necessary to correct for this when assessing these images. Figure 9 shows the both images of yarn’s planar projections, which are obtained from experiment. Figure 10 shows theoretical and experimental unwinding balloon profiles.

References


